2.29b Find $H_x(1,-2,3)$ if $H = g \circ p$,

where $p(x,y,z) = (x^3 - 2xyz, 3z + y^2 x)$

$g(x,y) = 4xy - x^2.$

$p : \mathbb{R}^3 \to \mathbb{R}^2$

$g : \mathbb{R}^2 \to \mathbb{R}.$

$H : \mathbb{R}^3 \to \mathbb{R}$

Chain Rule: $H'(x,y,z) = [g(p(x,y,z))]'$

$= g'(p(x,y,z)) \cdot p'(x,y,z)$

where $(x,y,z) = (1,-2,3)$

We have to calculate:

$p(1,-2,3) = (1^3 - 2(1)(-2)(3), 3(3) + 4 \cdot 1)$

$= (13, 13)$

$g' = \nabla g = (4y - 2x, 4x) \overset{(13,13)}{=}$

$(52 - 26, 52) = (26, 52)$

$p = (x^3 - 2xyz, 3z + y^2 x)$

$$p' = \begin{pmatrix} 3x^2 - 2yz, & -2xz, & -2xy \\ y^2, & 2yx, & 3 \end{pmatrix}$$

$$\overset{(1,-2,3)}{=} \begin{pmatrix} 3 - \cancel{2}(-6), & -6, & 4 \\ 4, & -4, & 3 \end{pmatrix} = \begin{pmatrix} 15 & -6 & 4 \\ 4 & -4 & 3 \end{pmatrix}$$

$$\Rightarrow H' = g'(p(x,y,z))\, p'(x,y,z)$$

$$= \begin{pmatrix} 26 & 52 \end{pmatrix} \begin{pmatrix} 15 & -6 & 4 \\ 4 & -4 & 3 \end{pmatrix}$$

$1 \times 2 \qquad 2 \times 3$

$$= (H_x, H_y, H_z)$$

$$= (26(15) + 52(4), \text{---}, \text{---})$$

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Example: Consider the function
$f(x,y) = 3x^2 + 3y^2$ in $\mathbb{R}^2$.
Find all directional derivatives of unit vectors at the point $(1,1)$.

Every unit vector is (v_1, v_2) with
$\sqrt{v_1^2+v_2^2} = 1 \Rightarrow v_1^2 + v_2^2 = 1$

ie $(v_1, v_2) = (\cos\theta, \sin\theta)$
$0 \le \theta \le 2\pi$
cover all possible unit vectors.

Our function is $f(x,y) = 3x^2 + 3y^2$

$$\nabla f = (6x, 6y) \overset{(1,1)}{=} (6,6).$$

$$\frac{\partial f}{\partial v} = \nabla f \cdot v = (6,6)\cdot\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$$

$v = (\cos\theta, \sin\theta)$

$$= 6\cos\theta + 6\sin\theta$$

Questions: ⓐ Where is $\frac{\partial f}{\partial v}$ maximum?

ⓑ Where is $\frac{\partial f}{\partial v}$ minimum?

ⓒ Where is $\frac{\partial f}{\partial v}$ 0?

"Where" - in what directions? What θ?

$$\frac{\partial f}{\partial v} = 6\cos\theta + 6\sin\theta$$

$$f(x,y) = 3x^2 + 3y^2$$

ⓐ For what θ is $\frac{\partial f}{\partial v} = 6\cos\theta + 6\sin\theta$ maximum?

$0 \le \theta < 2\pi$

$$\left(\frac{\partial f}{\partial v}\right)'(\theta) = -6\sin\theta + 6\cos\theta = 0$$

$$\Rightarrow \quad 6\sin\theta = 6\cos\theta$$

$$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

or $\theta = \frac{5\pi}{4}$.

possible extrema

θ	$\left(\frac{\partial f}{\partial v}\right)(\theta) = 6\cos\theta + 6\sin\theta$
0	$6\cos(0) + 6\sin(0) = 6$
2π	6
$\pi/4$	$6\frac{1}{\sqrt{2}} + 6\frac{1}{\sqrt{2}} = \frac{12}{\sqrt{2}} \approx 8.5$
$5\pi/4$	$6(-1/\sqrt{2}) + 6(-1/\sqrt{2}) = -\frac{12}{\sqrt{2}} \approx -8.5$

max when $\theta = \frac{\pi}{4}$

(b) min when $\theta = \frac{5\pi}{4}$

(c) when is $\frac{\partial f}{\partial v}(\theta) = 0$

$$6\cos\theta + 6\sin\theta = 0$$

$$\sin\theta = -\cos\theta$$

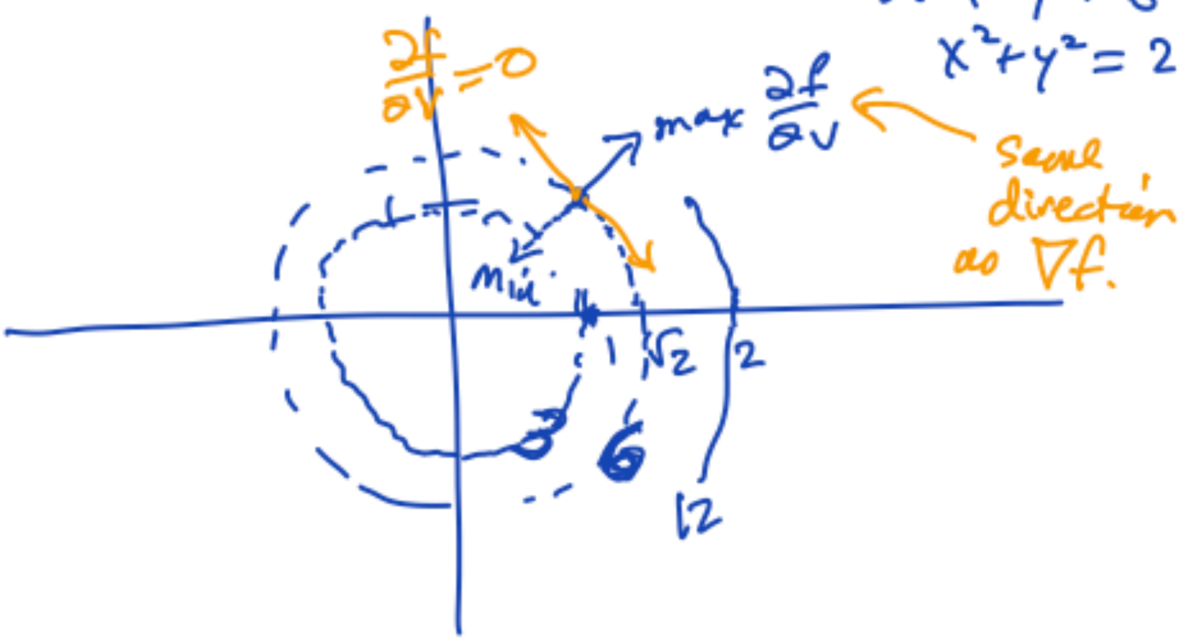
$$\tan\theta = -1$$

$$\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$$

$$f(x,y) = 3x^2 + 3y^2$$

$$3x^2 + 3y^2 = 6$$

$$x^2 + y^2 = 2$$



<u>New topic</u>: Finding Critical points, maxs & mins & saddles, for functions of several variables.

Related to: Finding absolute max & min of a fcn of several variables.

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1 To find <u>critical points</u>, we just find where all partial derivatives are 0 (or undefined) $\Longleftrightarrow$ Where the gradient is the zero vector.

<u>Example</u>: 1 Find the critical points of $g(x,y) = -x^3 + 4xy - 2y^2 + 7$

$\nabla g = (0, 0)$

$$\Rightarrow \left(-3x^2+4y,\ 4x-4y\right)$$
$$= (0,0)$$

$$-3x^2+4y=0$$
$$4x-4y=0$$

$$4x=4y \Rightarrow x=y$$

$$-3x^2+4x=0$$
$$x(-3x+4)=0$$
$$\Rightarrow x=0 \quad \text{OR} \quad -3x+4=0$$

$$\Downarrow \qquad\qquad\qquad 3x=4$$
$$y=0 \qquad\qquad\qquad x=4/3$$
$$\Downarrow$$
$$y=4/3$$

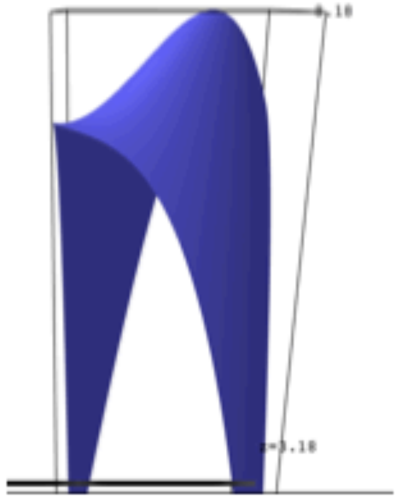
Two critical points

$$(0,0) \ \& \ (4/3,\ 4/3)$$

$$\uparrow \qquad \nearrow$$

If we graph $z=g(x,y)=-x^3+4xy-2y^2+7$, when $(x,y)=(0,0)$ or $\left(\frac{4}{3},\frac{4}{3}\right)$, the graph should have a horizontal tangent

plane (ie parallel to the xy plane).
Could be maxes, mins, or saddles.



Looks like $(0,0)$ is a saddle and $\left(\frac{4}{3}, \frac{4}{3}\right)$ is a local max of $g(x,y)$.

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(2) Find the critical points of

$$h(x,y) = \left(\tfrac{1}{2} - x^2 + y^2\right) e^{1-x^2-y^2}$$

$\nabla h = (0,0)$ or (undefined)

$= (h_x, h_y)$

$h_y = 2y e^{\sim} + \left(\tfrac{1}{2} - x^2 + y^2\right) e^{\sim}(-2y)$

$2y e^{\sim}\left(1 - \tfrac{1}{2} + x^2 - y^2\right)$

$$= \left(-2x e^{1-x^2-y^2}\left(1 + \left(\tfrac{1}{2} - x^2 + y^2\right)\right),\ 2y e^{1-x^2-y^2}\left(\tfrac{1}{2} + x^2 - y^2\right) \right)$$

$$= \left(-2x\left(\tfrac{3}{2} - x^2 + y^2\right) e^{1-x^2-y^2},\ 2y\left(\tfrac{1}{2} + x^2 - y^2\right) e^{1-x^2-y^2} \right)$$

$= (0, 0)$

$$\begin{cases} -2x\left(\frac{3}{2} - x^2 + y^2\right) \underbrace{e^{1-x^2-y^2}} = 0 \\ 2y\left(\frac{1}{2} + x^2 - y^2\right) \underbrace{e^{1-x^2-y^2}}_{\text{always } > 0} = 0 \end{cases}$$

divide by them

$\star$ $-2x\left(\frac{3}{2} - x^2 + y^2\right) = 0$

$\star\star$ $2y\left(\frac{1}{2} + x^2 - y^2\right) = 0$

$\star$ $-2x = 0$ OR $\frac{3}{2} - x^2 + y^2 = 0$

$\Rightarrow x = 0$ $\qquad x^2 - y^2 = \frac{3}{2}$

$\vdots$

finish on Thursday

$\Rightarrow \star\star$ $2y\left(\frac{1}{2} + 0 - y^2\right) = 0$

$\Rightarrow y = 0$ $\boxed{\text{OR}}$ $\frac{1}{2} - y^2 = 0$

$\Downarrow$ $\qquad y^2 = \frac{1}{2}$

$(0,0)$ $\qquad y = \pm \frac{1}{\sqrt{2}}$

$\left(0, \frac{1}{\sqrt{2}}\right)$

$\left(0, \frac{-1}{\sqrt{2}}\right)$